(12)

ប្រឡងជ្រើសរើសសិស្សពូកែទូទាំងខេត្តផ្នែកអក្សរសិល្ប៍ខ្មែរ គណិតវិទ្យា និងរូបវិទ្យា

ថ្នាក់ទី៩និង ទី១២ ឆ្នាំសិក្សា ២០០៩-២០១០

2010

សម័យប្រឡងថ្ងៃទី ២៣-០២-២០១០

វិញ្ញាសាទី២: គណិតវិទ្យាថ្នាក់ទី១២ (រយៈពេល២ម៉ោង-ពិន្ទុ១០០)

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1. គេឱ្យអនុគមន៍ $f(x)$ កំណត់លើចន្លោះ $[2;+\infty[$ ដោយ $f(x)=\dfrac{3x+Sinx}{x-1}$ ។
បង្ហាញថាចំពោះគ្រប់ $x\geq 2$, $|f(x)-3|\leq \dfrac{8}{x}$ ។ ទាញរកតម្លៃលីមីតនៃ $f(x)$ កាលណា x ខិតជិត $+\infty$ ។ (១០ពិន្ទុ)

2. ដោះស្រាយសមីការ : $2^{Cosx}=Cosx+\dfrac{1}{Cosx}$ ។ (១០ពិន្ទុ)

3. រកតួធំបំផុតនៃពន្លាតកន្សោម : $(\dfrac{9}{10}+\dfrac{1}{10})^{100}$ ។ (១៥ពិន្ទុ)

4. គេឱ្យកន្សោម : $\sum_1 = 1+Cos\theta+Cos2\theta+Cos3\theta+\cdots\cdots+Cos(n\theta)$

និង $\sum_2 = Sin\theta+Sin2\theta+Sin3\theta+\cdots\cdots+Sin(n\theta)$ ។

គណនាផលបូក: $S=\sum_1+i\sum_2$,រួចទាញរកតម្លៃនៃ $\sum_1$ និង $\sum_2$ ។ (២០ពិន្ទុ)

5. រង្វង់បីមានផ្ចិតរៀងគ្នា A ; B និង C , មានរង្វាស់កាំរៀងគ្នា p ; q និង r ,ប៉ះគ្នាពីរៗខាងក្រៅត្រង់ចំណុច D ; E និង F ។

បង្ហាញថាផលធៀប: $\dfrac{S(\Delta DEF)}{S(\Delta ABC)}=\dfrac{2pqr}{(p+q)(q+r)(r+p)}$ ។ (២០ពិន្ទុ)

($S(\Delta DEF)$:តាងផ្ទៃក្រលានៃត្រីកោណ DEF , $S(\Delta ABC)$:តាងផ្ទៃក្រលានៃត្រីកោណ ABC) ។

6. គេឱ្យស៊ីត (a_n) និងស៊ីត (b_n) ដែលកំណត់ដោយ :

$a_0=\dfrac{\sqrt{2}}{2}$; $a_{n+1}=\dfrac{\sqrt{2}}{2}\sqrt{1-\sqrt{1-a_n^2}}$ ដែល $n\geq 0$,និង $b_0=1$; $b_{n+1}=\dfrac{\sqrt{1+b_n^2}-1}{b_n}$ ដែល $n\geq 0$ ។

បង្ហាញថា: $2^{n+2}\cdot a_n<\pi<2^{n+2}\cdot b_n$ ។ (២៥ពិន្ទុ)

(កំណែ) 2010 . វិ II .

សន្លឹកទី២ ហត្ថលេខាគ្រូ

1/ ស្រាយបញ្ជាក់ : $f(x) - 3 = \frac{3x+\sin x}{x-1} - 3 = \frac{3+\sin x}{x(1-\frac{1}{x})}$

ចំពោះ:

- $-1 \le \sin x \le 1 \Rightarrow 2 \le 3+\sin x \le 4 \Rightarrow |3+\sin x| \le 4$ (1) 2
- $x \geq 2 \Rightarrow 0 < \frac{1}{x} \le \frac{1}{2} \Rightarrow 1-\frac{1}{x} \geq \frac{1}{2} \Rightarrow \frac{1}{x-1} \le \frac{2}{x}$ (2) 3

(1) x (2) អង្គនិងអង្គ : $\frac{|3+\sin x|}{x-1} \le 4 \times \frac{2}{x} = \frac{8}{x}$

ដូចនេះ $\boxed{|f(x)-3| \le \frac{8}{x} \text{ ចំពោះ } x \geq 2}$

$\lim_{x\to+\infty} |f(x)-3| \le \lim_{x\to+\infty} \frac{8}{x} = 0 \Rightarrow \lim_{x\to+\infty} f(x) = 3$ 5

2/ $2^{\cos x} > 0, \forall x \in \mathbb{R} \Rightarrow \cos x + \frac{1}{\cos x} > 0 \Rightarrow \cos x > 0$ 2

តាមវិសមភាព Cauchy :

- $\cos x + \frac{1}{\cos x} \geq 2\sqrt{\cos x \cdot \frac{1}{\cos x}} = 2$ (1) 2

និង $0 < \cos x \le 1 \Rightarrow 2^0 < 2^{\cos x} \le 2^1 = 2$ (2) 2

តាម (1) & (2) សមីការមានឫសលុះត្រាតែ : $2^{\cos x} = \cos x + \frac{1}{\cos x} = 2$ 2

$\cos x = 1 \Rightarrow \boxed{x = 2\pi k, \ k \in \mathbb{Z}}$ 2

3/ តាង: a_k តួទី k នៃពហុធា $\left(\frac{9}{10}+\frac{1}{10}\right)^{100}$, គេបាន: 2

$a_k = C(100,k)\left(\frac{9}{10}\right)^{100-k}\left(\frac{1}{10}\right)^k = \frac{100!}{(100-k)!\,k!}\cdot\frac{9^{100-k}}{10^{100-k}}\cdot\frac{1}{10^k}$

$= \frac{100!}{(100-k)!\,k!}\cdot\frac{9^{100-k}}{10^{100}}$ 3

$a_{k+1} = C(100,k+1)\left(\frac{9}{10}\right)^{100-k-1}\left(\frac{1}{10}\right)^{k+1} = \frac{100!}{(100-k-1)!\,(k+1)!}\cdot\frac{9^{100-k-1}}{10^{100}}$ 2

គេបាន $\frac{a_k}{a_{k+1}} = \frac{100!}{(100-k)!\,k!}\cdot\frac{9^{100-k}}{10^{100}}\times\frac{(100-k-1)!\,(k+1)!}{100!}\times\frac{10^{100}}{9^{100-k-1}} = \frac{9(k+1)}{100-k}$ 2

• បើ $\frac{a_k}{a_{k+1}} > 1 \Rightarrow \frac{9(k+1)}{100-k} > 1 \Rightarrow 9k+9 > 100-k$

$10k > 100-9 = 91 \Rightarrow k > \frac{91}{10} = 9,1$

ចំពោះ $k \geq 10 \Rightarrow a_k > a_{k+1}$ 2

• បើ $\frac{a_k}{a_{k+1}} < 1 \Rightarrow \frac{9(k+1)}{100-k} < 1 \Rightarrow 9k+9 < 100-k$

$10k < 100-9 = 91 \Rightarrow k < \frac{91}{10} = 9,1$ 2

ចំពោះ $k \leq 9 \Rightarrow a_k < a_{k+1}$

ដូចនេះ បើ $k \geq 10 \rightarrow a_{10} > a_{11} > a_{12} > \dots > a_{100}$
បើ $k \leq 9 \Rightarrow a_0 < a_1 < a_2 < \dots < a_{10}$ $\Bigg\} \Rightarrow a_{10}$ គឺជាតួមេគុណធំជាងគេ 2

ក្នុងពន្លាត $\left(\frac{9}{10}+\frac{1}{10}\right)^{100}$ ។

4/- គណនាផលបូក $S = \Sigma_1 + i\Sigma_2$ រួចទាញរកតម្លៃ Σ_1 និង Σ_2 :

យើងមាន: $\Sigma_1 = 1 + \cos\theta + \cos 2\theta + \dots + \cos(n\theta)$

$\Sigma_2 = \sin\theta + \sin 2\theta + \dots + \sin(n\theta)$

$S = \Sigma_1 + i\Sigma_2 = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \dots + \cos(n\theta) + i(\sin\theta + \sin 2\theta + \dots + \sin(n\theta))$

$= 1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \dots + (\cos(n\theta) + i\sin(n\theta))$ 5

$= 1 + (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta)^2 + \dots + (\cos\theta + i\sin\theta)^n$

S ជាផលបូក $(n+1)$ តួដំបូងនៃស្វ៊ីតធរណីមាត្រ ដែលមានតួទី1 $u_1 = 1$ និងផលធៀប :

$q = \cos\theta + i\sin\theta$ ។

នោះ: $S = \frac{(\cos\theta + i\sin\theta)^{n+1} - 1}{\cos\theta + i\sin\theta - 1} = \frac{1 - \cos(n+1)\theta - i\sin(n+1)\theta}{1 - \cos\theta - i\sin\theta}$ 2

$= \frac{(1 - \cos(n+1)\theta - i\sin(n+1)\theta)(1 - \cos\theta + i\sin\theta)}{(1 - \cos\theta - i\sin\theta)(1 - \cos\theta + i\sin\theta)}$

⊙ $(1 - \cos\theta - i\sin\theta)(1 - \cos\theta + i\sin\theta) = (1 - \cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$

$= 2 - 2\cos\theta = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}$ ។ 2

⊙ $(1 - \cos(n+1)\theta - i\sin(n+1)\theta)(1 - \cos\theta + i\sin\theta) =$

$= 1 - \cos\theta + i\sin\theta - \cos(n+1)\theta + \cos(n+1)\theta.\cos\theta - i\cos(n+1)\theta.\sin\theta - i\sin(n+1)\theta$
$+ i\sin(n+1)\theta.\cos\theta + \sin(n+1)\theta.\sin\theta$

$= (1 - \cos\theta - \cos(n+1)\theta + \cos(n+1)\theta.\cos\theta + \sin(n+1)\theta.\sin\theta) +$
$+ i(\sin\theta + \sin(n+1)\theta\cos\theta - \sin\theta\cos(n+1)\theta - \sin(n+1)\theta)$

$$= \left(1 - \cos\theta - \cos(n+1)\theta + \cos n\theta\right) + i\left(\sin\theta + \sin(n\theta) - \sin(n+1)\theta\right)$$

$$= \left(2\sin^2\frac{\theta}{2} + \cos n\theta - \cos(n+1)\theta\right) + i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 2\cos\frac{n\theta+(n+1)\theta}{2}\cdot\sin\frac{n\theta-(n+1)\theta}{2}\right)$$

$$= \left(2\sin^2\frac{\theta}{2} - 2\sin\frac{n\theta+(n+1)\theta}{2}\cdot\sin\frac{n\theta-(n+1)\theta}{2}\right) + i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 2\cos\frac{2n+1}{2}\theta\cdot\sin\frac{\theta}{2}\right)$$

$$= \left(2\sin^2\frac{\theta}{2} + 2\sin\frac{2n+1}{2}\theta\cdot\sin\frac{\theta}{2}\right) + i\,2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \cos\frac{2n+1}{2}\theta\right)$$

$$= 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \sin\frac{2n+1}{2}\theta\right) + i\,2\sin\frac{\theta}{2}\left(-2\sin\frac{n+1}{2}\theta\,\sin\left(-\frac{n\theta}{2}\right)\right)$$

$$= \left(2\sin\frac{\theta}{2}\cdot 2\sin\frac{n+1}{2}\theta\,\cos\frac{n\theta}{2}\right) + i\,4\sin\frac{\theta}{2}\cdot\sin\frac{n+1}{2}\theta\cdot\sin\frac{n\theta}{2}$$

$$= \left(4\sin\frac{\theta}{2}\cdot\sin\frac{n+1}{2}\theta\,\cos\frac{n\theta}{2}\right) + i\,4\sin\frac{\theta}{2}\cdot\sin\frac{n+1}{2}\theta\,\sin\frac{n\theta}{2}$$

5

ដោយជំនួស: $S = \dfrac{4\sin\frac{\theta}{2}\cdot\sin\frac{n+1}{2}\theta\cdot\cos\frac{n\theta}{2} + i\,4\sin\frac{\theta}{2}\sin\frac{n+1}{2}\theta\sin\frac{n\theta}{2}}{4\sin^2\frac{\theta}{2}}$

ដូចនេះ $\boxed{S = \dfrac{\sin(\frac{n+1}{2}\theta)\cdot\cos\frac{n\theta}{2}}{\sin\frac{\theta}{2}} + i\,\dfrac{\sin(\frac{n+1}{2}\theta)\cdot\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}}$ 2

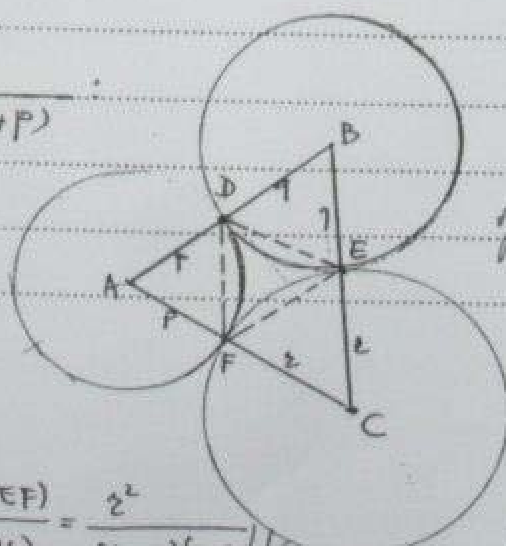
ទាញរកផលបូក Σ_1 និង Σ_2 :

តាមរូប:

$$\Sigma_1 = 1 + \cos\theta + \cos 2\theta + \cdots + \cos(n\theta) = \frac{\sin(\frac{n+1}{2})\theta\cdot\cos\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$ 2

$$\Sigma_2 = \sin\theta + \sin 2\theta + \cdots + \sin(n\theta) = \frac{\sin(\frac{n+1}{2}\theta)\cdot\sin(\frac{n\theta}{2})}{\sin\frac{\theta}{2}}$$ 2

5. បង្ហាញថាផលធៀបក្រឡាផ្ទៃ: $\dfrac{S(\Delta DEF)}{S(\Delta ABC)} = \dfrac{2p\cdot q\cdot r}{(p+q)(q+r)(r+p)}$:



ស្រាយបញ្ជាក់:

$S(\Delta ADF) = \frac{1}{2}p^2\sin\hat{A}$ 2

$S(\Delta ABC) = \frac{1}{2}(p+q)(p+r)\sin\hat{A}$ 2

នាំឲ្យ: $\dfrac{S(\Delta ADF)}{S(\Delta ABC)} = \dfrac{p^2}{(p+q)(p+r)}$ 2

ស្រាយដូចគ្នា: $\dfrac{S(\Delta BDE)}{S(\Delta ABC)} = \dfrac{q^2}{(p+q)(q+r)}$; $\dfrac{S(\Delta CEF)}{S(\Delta ABC)} = \dfrac{r^2}{(r+p)(q+r)}$

ពិន្ទុ 5

គេបាន: $\dfrac{S(\Delta DEF)}{S(\Delta ABC)} = \dfrac{S(\Delta ABC) - S(\Delta ADF) - S(\Delta BDE) - S(\Delta CEF)}{S(\Delta ABC)}$

$$= 1 - \frac{S(\Delta ADF)}{S(\Delta ABC)} - \frac{S(\Delta BDE)}{S(\Delta ABC)} - \frac{S(\Delta CEF)}{S(\Delta ABC)}$$

$$= 1 - \frac{p^2}{(p+q)(p+r)} - \frac{q^2}{(p+q)(q+r)} - \frac{r^2}{(p+r)(q+r)} = \frac{2pqr}{(p+q)(q+r)(r+p)}$$

ដូចនេះ: $\boxed{\dfrac{S(\Delta DEF)}{S(\Delta ABC)} = \dfrac{2pqr}{(p+q)(q+r)(r+p)}}$

6. បង្ហាញថា $2^{n+2} a_n < \pi < 2^{n+2} b_n$:

យើងមាន: $a_0 = \frac{\sqrt{2}}{2}$ និង $a_{n+1} = \frac{\sqrt{2}}{2}\sqrt{1-\sqrt{1-a_n^2}} \Rightarrow 0 < a_n < 1$ ចំពោះ: $\forall n \geq 0$ ។

យើងតាង: $\alpha_n \in \left[0, \frac{\pi}{2}\right]$ ដែល: $a_n = \sin(\alpha_n)$ និង $a_0 = \frac{\sqrt{2}}{2} = \sin\alpha_0 \Rightarrow \alpha_0 = \frac{\pi}{4} = \frac{\pi}{2^2}$

គេបាន: $a_{n+1} = \sin\alpha_{n+1} = \frac{\sqrt{2}}{2}\sqrt{1-\sqrt{1-\sin^2\alpha_n}} = \frac{\sqrt{2}}{2}\sqrt{1-\cos\alpha_n} = \frac{\sqrt{2}}{2}\sqrt{2\sin^2\frac{\alpha_n}{2}}$

$$= \sin\frac{\alpha_n}{2}$$

$a_{n+1} = \sin\alpha_{n+1} = \sin\frac{\alpha_n}{2} \Rightarrow \alpha_{n+1} = \frac{\alpha_n}{2}$

• បើ $n = 0 \Rightarrow \alpha_1 = \frac{\alpha_0}{2} = \frac{\pi/2^2}{2} = \frac{\pi}{2^{1+2}} = \frac{\pi}{2^3}$

• បើ $n = 1 \Rightarrow \alpha_2 = \frac{\alpha_1}{2} = \frac{\pi}{2^4} = \frac{\pi}{2^{2+2}}$ ទាញបាន $\Rightarrow \alpha_n = \frac{\pi}{2^{n+2}}$ (1)

យើងមាន:

$b_0 = 1$ និង $b_{n+1} = \dfrac{\sqrt{1+b_n^2}-1}{b_n} \Rightarrow b_n > 0$, ចំពោះ: $\forall n \geq 0$ ។

យើងតាង: $\beta_n \in \left[0, \frac{\pi}{2}\right]$ ដែល: $b_n = \tan(\beta_n)$ និង $b_0 = 1 = \tan\beta_0 \Rightarrow \beta_0 = \frac{\pi}{4} = \frac{\pi}{2^2}$ ។

គេបាន: $b_{n+1} = \tan\beta_{n+1} = \dfrac{\sqrt{1+\tan^2\beta_n}-1}{\tan\beta_n} = \dfrac{\frac{1}{\cos\beta_n}-1}{\sin\beta_n/\cos\beta_n} = \tan\frac{\beta_n}{2}$

$b_{n+1} = \tan\beta_{n+1} = \tan\frac{\beta_n}{2} \Rightarrow \beta_{n+1} = \frac{\beta_n}{2}$

• បើ $n = 0 \Rightarrow \beta_1 = \frac{\beta_0}{2} = \frac{\pi/4}{2} = \frac{\pi}{8} = \frac{\pi}{2^3} = \frac{\pi}{2^{1+2}}$

• បើ $n = 1 \Rightarrow \beta_2 = \frac{\beta_1}{2} = \frac{\pi}{2^4} = \frac{\pi}{2^{2+2}}$ ទាញបាន: $\beta_n = \frac{\pi}{2^{n+2}}$ (2)

តាម (1) និង (2): $\alpha_n = \beta_n = \frac{\pi}{2^{n+2}}$; $\forall n \in \mathbb{N}$,

ចំពោះគ្រប់ $x \in \left]0, \frac{\pi}{2}\right[$, គេបាន: $\sin\alpha_n < \alpha_n < \tan\alpha_n$

$a_n < \frac{\pi}{2^{n+2}} < b_n$ ដោយ $\alpha_n = \beta_n = \frac{\pi}{2^{n+2}}$

ដូចនេះ: $\boxed{2^{n+2} a_n < \pi < 2^{n+2} b_n}$